ON WAVES GENERATED ON THE SURFACE OF AN INCOMPRESSIBLE FLUID BY A SHOCK WAVE

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We consider the two- and three-dimensional problems of motion of a fluid resulting from pressure applied to its surface, neglecting the effect of gravity. We investigate a number of self-similar solutions.

When a blast occurs above the surface of a fluid (Fig. 1), then after a certain time the shock wave reaches the fluid and interacts with it. To determine the motion of the fluid and the gas, it is necessary to solve the problem simultaneously in both domains. However, considering the ratio of the densities of the two media, we may, as a first approximation, assume that displacements of the fluid do not influence the motion of the gas, which we suppose known. Such a formulation brings us to the problem of determining the motion of a fluid due to a pressure applied to its surface which varies according to a known law. The analogously-formulated linear problem for a compressible fluid is considered in the paper [1] but the author restricts himself to pressure fields. We suppose the fluid incompressible. This assumption is justified in the case of air and water for pressures resulting from shock waves and not exceeding 22 kg/cm.

1. The two-dimensional problem. We first consider the two-dimensional problem. In view of the fact that the motion starts from a state of equilibrium there exists a velocity potential ϕ which satisfies the equation

$$\triangle\varphi = \frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2} = 0 \quad (\nabla\varphi = -\mathbf{v}) \tag{1.1}$$

in the domain of flow and the boundary condition :

$$p(x, 0; t) = p_0(x, t) \quad \text{on the } x\text{-axis} \tag{1.2}$$

$$\partial\varphi / \partial n = 0 \quad \text{on the solid boundaries} \tag{1.3}$$

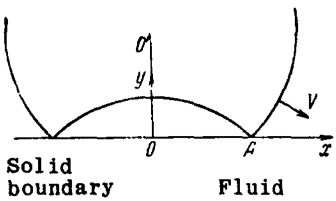

FIG. 1.

Here n is the direction of the normal to the solid boundary. If we use the Cauchy integral and ignore second-order terms and gravity, then condition (1.2) can be represented in the form

$$\frac{\partial \varphi}{\partial t} = p_0(x, t) \text{ on the } x\text{-axis} \tag{1.4}$$

We restrict ourselves to the consideration of three cases: a) The pressure due to the shock wave is constant; b) The pressure due to the shock wave is an arbitrary function of x/t; and c) The case of a cylindrical blast on the surface of the water.

a) Let a shock wave perpendicular to the wall CO travel along that wall with velocity V (Fig. 2). At O the shock wave encounters the free surface of a fluid of density ρ_1 and moves along that surface so that the point A travels in the direction of the positive x-axis with velocity $V \csc \alpha$. On the segment AB the pressure may be assumed to be zero. On OA the pressure is determined from the solution of the gasdynamical problem which is supposed known [2,3]. The problem under consideration has self-similar solutions since the flow depends only on the dimensionless combinations $\xi = x/Vt$, $\eta = y/Vt$ and the angles α and β.

In dimensionless coordinates the relations (1.1), (1.3) and (1.4) take the form

$$\frac{\partial^2 \Phi}{\partial \xi^2} + \frac{\partial^2 \Phi}{\partial \eta^2} = 0 \qquad (\varphi = \Phi(\xi, \eta) V^2 t) \tag{1.5}$$

$$\begin{aligned} &\Phi - \left(\xi \frac{\partial \Phi}{\partial \xi} + \eta \frac{\partial \Phi}{\partial \eta}\right) = a(\xi) \qquad \text{for } 0 < \xi < \frac{1}{\varepsilon} \qquad \left(a = \frac{P_0(\xi)}{V^2 \rho_1}\right) \\ &\Phi - \left(\xi \frac{\partial \Phi}{\partial \xi} + \eta \frac{\partial \Phi}{\partial \eta}\right) = 0 \qquad \text{for } \frac{1}{\varepsilon} < \xi < \infty \qquad (\varepsilon = \sin \alpha) \end{aligned} \tag{1.6}$$

$$\frac{\partial \Phi}{\partial n_1} = 0 \quad \text{on the solid boundary} \tag{1.7}$$

If we go over to complex variables by putting $w = \Phi + i\Psi$, $z = \xi + i\eta$, then the boundary conditions (1.6) become

$$\begin{aligned} &\mathrm{Re}\,(w-\zeta dw/d\zeta)=a_1(\zeta_1) &&\text{for } 0<\zeta_1<1 \quad (\zeta=z\varepsilon) \\ &\mathrm{Re}\,(w-\zeta dw/d\zeta)=0 &&\text{for } 1<\zeta_1<\infty \ (\zeta_1=\xi\varepsilon) \end{aligned} \tag{1.8}$$

We introduce a new analytic function $W = w - \zeta dw/d\zeta$. For this function the boundary conditions (1.6) take the form

$$\mathrm{Re}\,W = a_1(\zeta_1) \quad \text{for} \quad 0<\zeta_1<1, \qquad \mathrm{Re}\,W=0 \quad \text{for } 1<\zeta_1<\infty \tag{1.9}$$

Finally we introduce the variable $\tau = \zeta^k$ which varies in the lower-right quadrant where $k = \pi/2\beta$. Let $W = W_1(\tau)$. Using the reflection principle we continue the function $W_1(\tau)$ into the lower-left quadrant. As a result of symmetry relative to the η_1-axis, condition (1.7) is automatically satisfied. We can now determine W_1 from Schwarz's formula

$$W_1 = -\frac{1}{\pi i}\int_{-1}^{1} a_1\left(t^{\frac{1}{k}}\right)\frac{dt}{t-\tau} \tag{1.10}$$

Since in our case $a_1 = \text{const}$, W has the form

$$W = -\frac{a}{\pi i}\ln\frac{\zeta^k-1}{\zeta^k+1}$$

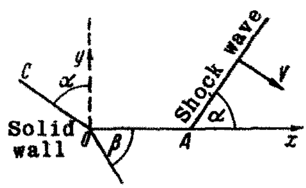

FIG. 2.

To find the function $dw/d\zeta = \epsilon^{-1}(-u+iv)$, where u and v are the horizontal and vertical velocity components respectively, we must integrate the equation

$$\zeta\frac{dw}{d\zeta} - w + W = 0 \tag{1.11}$$

After integration we get

$$\frac{dw}{d\zeta} = \frac{2ka}{\pi i}\int\frac{\zeta^{k-2}d\zeta}{\zeta^{2k}-1} \tag{1.12}$$

For $k = 1$, this integral becomes

$$\frac{dw}{d\zeta} = -\frac{a}{\pi i}\ln\frac{\zeta^2}{\zeta^2-1} \tag{1.13}$$

If k is an integer greater than unity we can get the following general formula:

$$\frac{dw}{d\zeta} = -\frac{2a}{\pi i}\left\{\frac{1}{2}[(-1)^k\ln(1+\zeta)-\ln(1-\zeta)] - \right. \tag{1.14}$$

$$-\frac{1}{2}\sum_{j=1}^{k-1}\cos\frac{j(k-1)\pi}{k}\ln\left[1-2\zeta\cos\frac{j\pi}{k}+\zeta^2\right]+$$

$$+\sum_{j=1}^{k-1} \sin \frac{j(k-1)\pi}{k} \operatorname{arc\,tg} \frac{\zeta - \cos\frac{j\pi}{k}}{\sin\frac{j\pi}{k}}\Bigg\} + C_k$$

where C_k are integration constants to be determined from the condition of equilibrium at ∞. For $k = 2$ and $k = 3$ we get

$$\frac{dw}{d\zeta} = -\frac{a}{\pi i}\left(\ln\frac{1+\zeta}{1-\zeta} + 2\tan^{-1}\zeta - \pi + \pi i\right) \quad \text{for } k = 2 \tag{1.15}$$

$$\frac{dw}{d\zeta} = -\frac{a}{\pi i}\left[\frac{1}{2}\ln\frac{(1-\zeta+\zeta^2)(1+\zeta+\zeta^2)}{(1-\zeta^2)^2} + \right.$$

$$\left. + \sqrt{3}\left(\operatorname{arc\,tg}\frac{2\zeta-1}{\sqrt{3}} - \tan^{-1}\frac{2\zeta+1}{\sqrt{3}}\right) + \pi i\right] \quad \text{for } k = 3 \tag{1.16}$$

We now compute the velocity at the origin. Equations (1.13), (1.15), and (1.16) imply

$$(u + iv)_{z=0} = a\varepsilon\left(1 - i\operatorname{tg}\frac{\pi}{2k}\right) = a\varepsilon(1 - i\operatorname{tg}\beta) \tag{1.17}$$

Thus, the velocity at the origin is directed along the solid boundary. From (1.13), (1.15) and (1.16) we see that the horizontal velocity u is equal to $a\epsilon$ in the high-pressure region and to zero in the region of zero pressure. The graphs of the dependence of the vertical velocity $v_1 = a^{-1}\,\mathrm{Im}\, dw/d\zeta$ on ζ_1 for $k = 1, 2, 3$ are shown in Fig. 3.

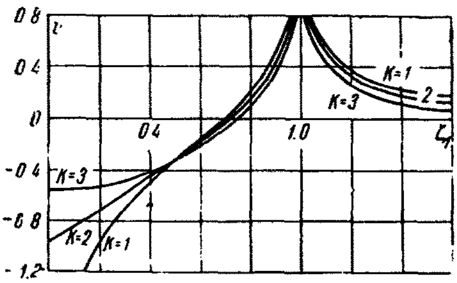

FIG. 3.

The singularities at the origin for $k = 1$ and at $\zeta = 1$ for all values of k reflect the shortcomings of the linear theory.

We compute the displacement S_1 of the particles of the fluid

$$S_1 = V\int_0^t (u + iv)\,dt \tag{1.18}$$

If we express u and v in terms of ζ and compute (1.18) we obtain the particle-displacement field for the fluid. We carry out the investigation only for points on the surface of the fluid. Clearly, the horizontal displacement is equal to $a\epsilon Vt$ in the region of high pressure and equal to zero in the region of zero pressure.

We consider the vertical displacement S_*. Integrating (1.18) and introducing the dimensionless distance $S = S_*\pi/a\epsilon Vt$ we get

$$S = -i\left(\ln\frac{\zeta_1^2-1}{\zeta_1^2} - 2 + \zeta_1\ln\frac{1+\zeta_1}{1-\zeta_1}\right) \qquad \text{for } k=1 \qquad (1.19)$$

$$S = i\left(\ln\frac{1+\zeta_1}{1-\zeta_1} + \zeta_1\ln\frac{1-\zeta_1^4}{\zeta_1^4} + 2\tan^{-1}\zeta_1 - \pi\right) \qquad \text{for } k=2 \qquad (1.20)$$

The graphs of the functions $S(\zeta_1)$ corresponding to these formulas are shown in Fig. 4. We note that for $k > 1$ the displacement field has no singularities.

b) When computing the velocity field we assumed that the pressure behind the shock wave was constant. This assumption is valid for large and small incidence angles a (cf. Fig. 2), i.e. for $a \approx 0$ and $a \approx \pi/2$. For intermediate values of a this is only approximately correct (thus, it is shown in [4] that when the incidence angle of the shock wave is small, the increased pressure is concentrated in a narrow region adjoining the shock wave). For a more accurate solution the function $a_1(\zeta_1)$

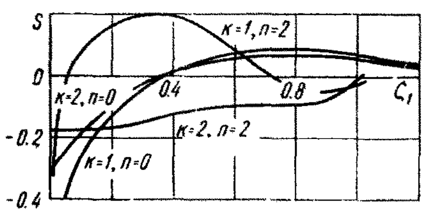

FIG. 4.

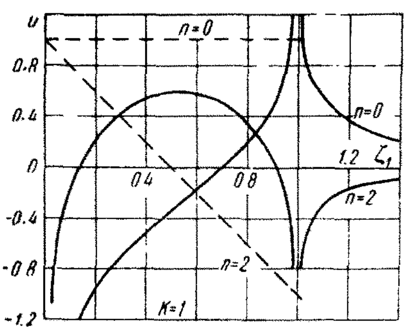

FIG. 5.

which gives the distribution of pressure on the surface of the liquid must be determined experimentally, or it can be assumed to be known from the solution of the gasdynamical problem of the reflection of a shock wave from an arbitrary angle. We assume the function $a_1(\zeta_1)$ to be known and we represent it as a power series in ζ_1

$$a_1(\zeta_1) = \sum_{n=0}^{\infty} a_n\zeta_1^n \qquad (1.21)$$

Substituting (1.21) in (1.10) we get

$$W = -\frac{1}{\pi i} \sum_{n=0}^{\infty} a_n \int_{-1}^{1} t^{\frac{n}{k}} \frac{dt}{t - \tau} \tag{1.22}$$

Substituting (1.22) in (1.11) and differentiating under the integral sign we find

$$\frac{dw}{d\zeta} = -\frac{k^2}{\pi i} \int \zeta^{k-2} \sum_{n=0}^{\infty} a_n \int_{-1}^{1} \frac{\zeta_1^{n+k-1} d\zeta_1}{(\zeta_1^k - \zeta^k)^2} d\zeta \tag{1.23}$$

The constant of integration is determined from the condition of equilibrium at infinity. For arbitrary integer values of k the inner integral leads to elementary functions. Thus, computation of the velocity field for an arbitrary function $a_1(\zeta_1)$ reduces to quadratures.

By way of example we consider one of the simplest cases of non-uniform distribution of pressure $a_1(\zeta_1) = a_2\zeta_1^2$, $a_2 = \text{const}$. Here, the greater part of the pressure is concentrated in the vicinity of the shock wave. Computing the integrals we get

$$\frac{dw}{d\zeta} = -\frac{a_2}{\pi i}\left[\ln \frac{\zeta^2}{\zeta^2 - 1} + 2\left(\zeta \ln \frac{\zeta - 1}{\zeta + 1} + 2\right)\right] \qquad \text{for } k = 1 \tag{1.24}$$

$$\frac{dw}{d\zeta} = -\frac{2a_2}{\pi i}\left(\zeta \ln \frac{\zeta^2 - 1}{\zeta^2 + 1} + \frac{1}{2} \ln \frac{\zeta + 1}{\zeta - 1} - \tan^{-1} \zeta + \frac{\pi}{2}\right) \qquad \text{for } k = 2 \tag{1.25}$$

Figure 5 gives the distributions of vertical (solid lines) and horizontal (broken lines) velocities computed from (1.24) and, for comparison, the corresponding curves in the case of uniform distribution of pressure.

Substituting (1.24) and (1.25) in (1.18) we get the vertical displacement of the surface

$$S = -i\left(\ln \frac{\zeta_1^2 - 1}{\zeta_1^2} - 6 + \zeta_1 \ln \frac{1 + \zeta_1}{1 - \zeta_1} + \pi^2\zeta_1 - 2\int_0^{\zeta_1} \ln \frac{1 + \zeta_1}{1 - \zeta_1} \frac{d\zeta_1}{\zeta_1}\right) \qquad \text{for } k = 1 \tag{1.26}$$

$$S = i\left(\ln \frac{1 + \zeta_1}{1 - \zeta_1} + \zeta_1 \ln \frac{1 - \zeta_1^2}{1 + \zeta_1^2} - 2\tan^{-1} \zeta_1 - \frac{\pi}{2}(\pi - 2) - \int_0^{\zeta_1} \ln \frac{1 - \zeta_1^2}{1 + \zeta_1^2} \frac{d\zeta_1}{\zeta_1}\right) \qquad \text{for } k = 2 \tag{1.27}$$

The graphs of these functions appear in Fig. 4. Using Formulas (1.24) and (1.25) it is easy to compute the horizontal displacement S'. S' is the same in both cases and has the value

$$S' = S_1' / Vt = a_2\varepsilon(1 - 2\zeta_1) \quad \text{for } 0 < \zeta_1 < 1, \qquad S' = 0 \quad \text{for } 1 < \zeta_1 < \infty$$

c) We assume that a cylindrical region of pressure is propagated outward from the point O. The pressure on the surface of the fluid is supposed to vary in the manner corresponding to a strong cylindrical blast with energy (1/2) E. In this case the flow depends on the parameters γ, ρ_1, ρ_2, E, x, y, t, where γ is the ratio of specific heats for the gas, ρ_1 is the density of the fluid and ρ_2 is the density of the gas. According to dimension theory the flow will depend on the dimensionless parameters

$$\xi = \frac{\rho_2^{1/4} x}{E^{1/4} t^{1/2}}, \qquad \eta = \frac{\rho_2^{1/4} y}{E^{1/4} t^{1/2}}, \qquad \gamma, \qquad \frac{\rho_2}{\rho_1}$$

We shall restrict ourselves to the most important case of a fluid of infinite depth. Equation (1.5) holds in the lower half-plane occupied by the fluid with the velocity potential ϕ given by

$$\varphi(x, y; t) = \Phi(\xi, \eta) \sqrt{\frac{E}{\rho_2}}$$

In dimensionless coordinates the boundary condition (1.4) on the surface takes the form

$$\xi \frac{\partial \Phi}{\partial \xi} + \eta \frac{\partial \Phi}{\partial \eta} = -\frac{h(\xi)}{\gamma + 1} \frac{\rho_2}{\rho_1} = -a_1(\xi) \qquad \left(h(\xi) = \frac{p}{p_2}\right) \tag{1.28}$$

where $h(\xi)$ is a known function [5] which describes the distribution of pressure in dimensionless coordinates in case of a strong blast and p_2 is the pressure behind a strong shock wave. If we introduce the complex variables $w = \Phi + i\Psi$, $z = \xi + i\eta$, then condition (1.28) becomes

$$\operatorname{Re}\left(z \frac{dw}{dz}\right) = -a_1(\xi) \quad \text{for } |\xi| < 1, \qquad \operatorname{Re}\left(z \frac{dw}{dz}\right) = 0 \quad \text{for } |\xi| > 1 \tag{1.29}$$

Thus, to determine the function $z\,dw/dz$ it is necessary to solve the Dirichlet problem. Using the Schwarz integral we find

$$\frac{dw}{dz} = \frac{1}{\pi i z} \int_{-1}^{1} a_1(t) \frac{dt}{t - z} \tag{1.30}$$

This formula describes the velocity field. To get an approximate idea of this field we take $a_1(t) = a$, a constant, which implies an error only in the vicinity of the shock wave. Then (1.30) yields

$$\frac{dw}{dz} = \frac{a}{\pi i} \frac{1}{z} \ln \frac{z - 1}{z + 1} \tag{1.31}$$

Using the approximate formula (1.31) it is easy to compute the displacement field. The result is

$$S = \frac{\pi S_1 \rho_2}{2 a E^{1/4} t^{1/2}} = -i\left(\frac{1 - z^2}{z} \ln \frac{z - 1}{z + 1} - 2\right) \tag{1.32}$$

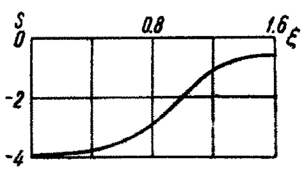

FIG. 6.

The graph of the vertical displacements of the particles of the surface of the fluid based on this formula is given in Fig. 6. For horizontal displacements we get

$$S' = a\frac{1-\xi^2}{\xi} \quad \text{for } |\xi| < 1, \qquad S' = 0 \quad \text{for } |\xi| > 1$$

We now compute the energy E_1 acquired by the fluid as a result of the blast. Using the approximate formula (1.31) we obtain

$$E_1 = \frac{\rho_1}{\rho_2} E \int_0^\infty \varphi \frac{d\varphi}{d\eta} d\xi = \frac{0.084}{(\gamma+1)^2} \frac{\rho_2}{\rho_1} E \tag{1.33}$$

Thus, the magnitude of the energy imparted to the fluid is proportional to the ratio of the densities of the two media.

2. The three-dimensional problem. Without changing our assumptions we consider the problem of motion of a fluid which fills the lower half-space under pressure due to a point blast in the gas. In the lower half-space the velocity potential satisfies the Laplace equation. Assuming axial symmetry the condition on the surface expressed in polar coordinates takes the form

$$\frac{\partial\varphi}{\partial t} = p_1(\rho, t) \qquad \text{for } z = 0 \tag{2.1}$$

where $p_1(\rho, t)$ is a known function. Let $\Phi(\rho, z; t) = \partial\phi/\partial t$. Obviously, this function must also satisfy Laplace's equation. Consider the zero order Hankel transform Φ° of Φ

$$\Phi^\circ(\xi, z; t) = \int_0^\infty r\Phi(r, z; t) J_0(\xi r)\, dr$$

Multiplying both sides of the Laplace equation by $rJ_0(\xi r)$ and integrating with respect to r from 0 to ∞ we arrive at the well-known conclusion that Φ° satisfies the equation

$$\frac{d^2\Phi^\circ}{dz^2} - \xi^2\Phi^\circ = 0$$

whose solution suitable in cases when $\Phi \to 0$ as $z \to -\infty$ has the form

$$\Phi^\circ = A(\xi, t) e^{\xi z}$$

Multiplying (2.1) by $rJ_0(\xi r)$, integrating with respect to r and putting $z = 0$ in the latter relation we find $A(\xi, t)$ and then using the inversion formula for the Hankel transform we get

$$\frac{\partial\varphi}{\partial t} = \int_0^\infty \xi J_0(\xi\rho) e^{\xi z} \int_0^\infty rJ_0(\xi r) p_1(r, t)\, dr d\xi \tag{2.2}$$

Formula (2.2) gives the pressure field in a fluid due to an arbitrary surface pressure $p_1(\rho, t)$. We now consider special cases.

a) Let the function $p_1(\rho, t)$ have the form

$$p_1 = a_1 \quad \text{for } \rho < Vt, \qquad p_1 = 0 \quad \text{for } \rho > Vt \qquad (a_1, V = \text{const})$$

This corresponds to a circular region of constant pressure which is propagated in all directions with constant velocity. In this formulation the problem is one with self-similar solutions, all flow parameters depending on the dimensionless quantities

$$a = \frac{a_1}{\rho_1 V^2}, \qquad \rho_1 = \frac{\rho}{Vt}, \qquad z_1 = \frac{z}{Vt}$$

In dimensionless variables the velocity potential takes the form

$$\varphi(\rho, z; t) = V^2 t \Phi(\rho_1, z_1)$$

The dimensionless form of Equation (2.2) is

$$\Phi - \rho_1 \frac{\partial \Phi}{\partial \rho_1} - z_1 \frac{\partial \Phi}{\partial z_1} = a \int_0^\infty e^{\xi z_1} J_0(\xi \rho_1) J_1(\xi)\, d\xi \tag{2.3}$$

We now expand $\Phi(\rho_1, z_1)$ in powers of z_1 but restrict ourselves to the first two terms

$$\Phi(\rho_1, z_1) = \Phi_0(\rho_1) + z_1 \Phi_1(\rho_1) + \ldots \tag{2.4}$$

Substituting (2.4) in (2.3), expanding the right side of (2.3) in a series of powers of z_1, and equating coefficients, we obtain the following expressions for the terms of order zero and one:

$$\Phi_0 - \rho_1 \frac{d\Phi_0}{d\rho_1} = a \int_0^\infty J_0(\xi \rho_1) J_1(\xi)\, d\xi \tag{2.5}$$

$$\rho_1 \frac{d\Phi_1}{d\rho_1} = -a \int_0^\infty \xi J_0(\xi \rho_1) J_1(\xi)\, d\xi \tag{2.6}$$

From (2.5) we find the horizontal velocity of the particles of the surface of the fluid

$$u = \frac{d\Phi_0}{d\rho_1} = a \quad \text{for } \rho_1 < 1, \qquad u = 0 \quad \text{for } \rho_1 > 1 \tag{2.7}$$

From (2.6) we find the corresponding vertical velocity

$$v_1 = \Phi_1(\rho_1) \frac{\pi^2}{2a} = -\left[\int_\infty^{\rho_1} \frac{3 - \rho^2}{(1 + \rho^2)^2 \sqrt{\rho}} \cos^2 \frac{1 + \rho^2}{2\rho}\, d\rho + \int_\infty^{\rho_1} \frac{1 - \rho^2}{(1 + \rho^2)\rho\sqrt{\rho}} \sin \frac{1 + \rho^2}{\rho}\, d\rho\right] \tag{2.8}$$

The arbitrary constant is chosen using the condition of equilibrium at infinity. The graph of vertical velocity based on this formula is given in Fig. 7 (the dotted curve). We now compute the displacement of the points on the surface of the fluid. We denote by S the dimensionless distance $S = S_1\pi^2/Vt\,2a$, where S_1 is the dimensional quantity. Further, we denote by S^1 and S_* the horizontal and vertical displacements. Using (2.8) and (1.18) we get for the horizontal displacement

$$S' = a \quad \text{for } \rho_1 < 1, \qquad S' = 0 \quad \text{for } \rho_1 > 1$$

The graph of vertical displacement based on (2.8) and (1.18) appears as the dotted curve in Fig. 8.

b) We now assume that the distribution of pressure $p_0(\rho, t)$ is such as would result from a strong blast at the point O on the surface of the fluid with $E/2$ the amount of energy imparted to the gas. In this formulation the problem is likewise self-similar with the flow depending on the dimensionless combinations

$$\rho_1 = \frac{\rho}{(E/\rho_2)^{1/5} t^{2/5}}, \qquad z_1 = \frac{z}{(E/\rho_2)^{1/5} t^{2/5}}, \qquad \gamma, \qquad \frac{\rho_2}{\rho_1}$$

The velocity potential ϕ is expressed in terms of the corresponding dimensionless function in the following manner:

$$\varphi(\rho, z; t) = \left(\frac{E}{\rho_2}\right)^{2/5} \frac{1}{t^{1/5}} \Phi(\rho_1, z_1)$$

The dimensionless form of Equation (2.2) becomes in this case

$$\Phi + 2\left(\rho_1 \frac{\partial\Phi}{\partial\rho_1} + z_1 \frac{\partial\Phi}{\partial z_1}\right) = a \int_0^\infty \xi J_0(\xi\rho_1) e^{\xi z_1} \int_0^1 r J_0(\xi r) h(r)\, dr d\xi \tag{2.9}$$

$$\left(a = -\frac{8}{5(\gamma+1)} \frac{\rho_2}{\rho_1}\right)$$

Here $h(r)$ is a known function which gives the distribution of pressure for a strong blast [5]. Its graph appears in Fig. 9. When $h(r)$ is a complicated function, the inner integral is not expressed in terms of elementary functions.

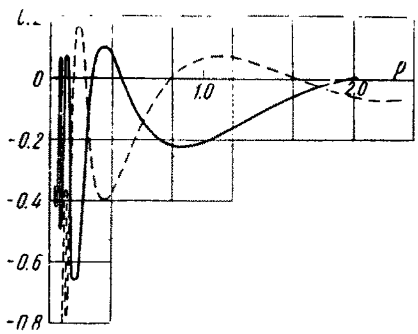

FIG. 7.

As can be seen from Fig. 9, $h(r)$ can be well approximated by putting $h(r) = b = 0.366$. Then (2.9) becomes

$$\Phi + 2\left(\rho_1 \frac{\partial\Phi}{\partial\rho_1} + z_1 \frac{\partial\Phi}{\partial z_1}\right) = = ab \int_0^\infty e^{\xi z_1} J_0(\xi\rho_1) J_1(\xi)\, d\xi \tag{2.10}$$

As in the previous example we obtain a differential equation for Φ_0 and Φ_1 by substituting (2.4) in (2.10), expanding the integral in a series and equating the coefficients of like powers of z_1. Solving these differential equations we get for the horizontal velocity $u = d\Phi_0/d\rho_1$ on the surface of the fluid

$$u = \frac{ab}{2\rho_1^{3/2}} \quad \text{for } \rho_1 < 1, \qquad u = 0 \quad \text{for } \rho_1 > 1$$

and for the vertical velocity $v_1 = \Phi_1\pi^2/ab$.

$$v_1 = -\left[\int_0^{\rho_1} \frac{(3-\rho^2)\rho}{(1+\rho^2)^2}\cos^2\frac{1+\rho^2}{2\rho}\,d\rho + \int_0^{\rho_1} \frac{1-\rho^2}{1+\rho^2}\sin\frac{1+\rho^2}{\rho}\,d\rho\right] + \frac{C}{\rho^{3/2}} \tag{2.11}$$

The arbitrary constant C is determined from the condition of finite energy imparted to the fluid. We find that $C = 0$. The graph of the vertical velocity v_1 appears as the solid curve in Fig. 7.

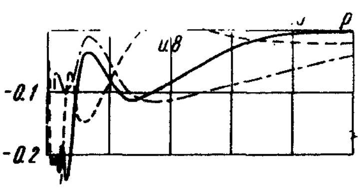

FIG. 8.

We introduce the dimensionless distance

$$S = \frac{2}{5} S_1 \left(\frac{E}{\rho_2}\right)^{-1/5} t^{-2/5}$$

The solid curve in Fig. 8 is the graph of the dimensionless vertical displacement obtained by numerical integration using (2.11) and (1.18).

The graph in Fig. 9 shows that the line $h(r) = b$ deviates strongly from the curve in the vicinity of the shock wave where on a small interval the pressure takes on large values. One can take this fact into

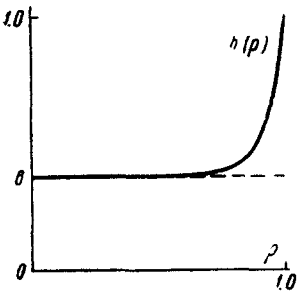

FIG. 9.

consideration and so make the theory just developed more precise; that is, we can assume that the function $h_1(r) = h(r) - b$ is of the form

$$h_1(r) = c\delta(r-1), \qquad c = \int_0^1 [h(r) - b]\,dr$$

where $\delta(r - 1)$ is the Dirac delta function.

Let Φ be the velocity potential of the required motion. In view of the linearity of the problem

$$\Phi = \Phi_b + \Phi_\delta$$

where Φ_b stands for the solution of the flow problem with uniform pressure and Φ_δ for the velocity potential of a flow due to a ringlike pressure zone. Since Φ_b was computed earlier, it remains to determine Φ_δ.

To this end we substitute $h_1(r)$ in (2.9) and obtain

$$\Phi_\delta + 2\left(\rho_1 \frac{\partial \Phi_\delta}{\partial \rho_1} + z_1 \frac{\partial \Phi_\delta}{\partial z_1}\right) = ac \int_0^\infty \xi e^{\xi z_1} J_0(\xi \rho_1) J_0(\xi) d\xi \qquad (2.12)$$

In an analogous manner we obtain the following expressions for the horizontal and vertical velocities:

$$u_\delta = 0$$

$$v_\delta = -\frac{4ac}{\pi \rho_1^{3/2}} \left[\int_0^{\rho_1} \frac{\rho}{(1+\rho^2)^2} \cos^2 \frac{1+\rho^2}{2\rho} d\rho + \int_0^{\rho_1} \frac{\rho}{2(1+\rho^2)} \sin \frac{1+\rho^2}{\rho} d\rho\right] \qquad (2.13)$$

Using Formulas (2.13) and (1.18) we obtain the corresponding vertical displacement. Its graph (multiplied by a factor of b/c) appears as the dotted curve in Fig. 8.

In case of a point blast it is possible to compute the energy E_1 imparted to the fluid. Using (2.11) we obtain, after computing the integral, the following formula

$$E_1 = \frac{0.0022}{(\gamma + 1)^2} \frac{\rho_2}{\rho_1} E$$

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